# ON A THEOREM OF BAZILEVIC FOR AREALLY MEAN $p$-VALENT FUNCTIONS ATTAINING MAXIMAL GROWTH ON $k$ RAYS*

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ABSTRACT

Let $\mu(p,k)$ denote the class of areally mean $p$-valent functions attaining maximal growth on $k$ rays. The aim of this article is to get the sufficient and necessary condition for which Bazilevic's theorem holds for $f \in \mu(p,k)$.

## 1. Introduction

Let $\mu(p)$ denote the class of functions $f$ which are areally mean $p$-valent in the unit disk $D = \{z\colon |z| < 1\}$. For integer $k$, we say $f$ is in $\mu(p,k)$ (see [5, p. 148]) if $f \in \mu(p)$ and we can find constants $c > 0$, $\delta > 0$ and a sequence $\{r_n\}$ with $r_n \to 1^-$ as $n \to \infty$ such that

$$(1.1)\qquad |f(r_n e^{i\theta_n^{(s)}})| \geq c(1-r_n)^{-2p/k}, \quad s = 1, 2, \ldots, k,$$

and

$$(1.2)\qquad \delta \leq |\theta_n^{(j)} - \theta_n^{(s)}| < 2\pi - \delta, \quad 1 \leq j < s \leq k,$$

for all $n$. By taking subsequences we may assume $e^{i\theta_n^{(s)}} \to e^{i\theta_s}$ as $n \to \infty$, $s = 1, 2, \ldots, k$. For $f \in \mu(p,k)$ we say that $f$ attains maximal growth on the $k$ rays $e^{i\theta_s}$, $s = 1, 2, \ldots, k$.

Let $M(r) = \max_{|z|=r} |f(z)|$. If $f \in \mu(p)$ and if $\lim_{r\to 1^-}(1-r)^{2p}M(r) = \alpha > 0$, then $f \in \mu(p,1)$.

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Let $z_1, \ldots, z_q$ be the zeros of $f(z)$ in $D$. For an appropriate constant $c \neq 0$ put $h(z) = c\prod_{j=1}^{q}(z - z_j)$ so that

(1.3) $$\log \frac{f(z)}{h(z)} = 2p\sum_{n=1}^{\infty} c_n z^n$$

is regular in $D$. The following theorem due to Aharonov [1] is a generalization of Bazilevic's Theorem [2].

THEOREM A: *Suppose that $f \in \mu(p, 1)$ and attains maximal growth on $e^{i\theta_0}$ and that*

(1.4) $$|f(z)| \geq A > 0, \quad 0 < r_0 \leq |z| < 1,$$

*for some constants $A$ and $r_0$. Then*

(1.5) $$\sum_{n=1}^{\infty} n\left|c_n - \frac{1}{n}e^{-i\theta_0}\right|^2 < +\infty,$$

*where $\{c_n\}$ are given by* (1.3).

Since Bazilevic's Theorem is important, it is natural to ask what is the sufficient and necessary condition for which (1.5) holds. In the present paper we shall solve this problem. Moreover, we answer this question for the class $\mu(p, k)$.

## 2. Another definition of the class $\mu(p, k)$

Let the simply connected regions $D_1, \ldots, D_k$ be disjoint and contained in $D$, and furthermore satisfy (i) $e^{i\theta_s} \in \partial D_s$, $s = 1, 2, \ldots, k$, (ii) $\bar{D}_1 \cup \bar{D}_2 \cup \cdots \cup \bar{D}_k = \bar{D}$, (iii) for each $j$, the function

$$t(z) = \frac{k}{2}\log\frac{z}{(1 - ze^{-i\theta_1})^{2/k}\cdots(1 - ze^{-i\theta_k})^{2/k}}$$

maps $D_j$ one to one conformally onto the strips $S_j = \{t\colon b_j < \operatorname{Im} t < b_j + \pi\}$, where $b_j$ is real. The possibility follows from the argument in [7, p. 44]. The inverse mapping $z = \varphi_j(t)$ maps $S_j$ onto $D_j$.

Consider the function $g_j(t) = f(\varphi_j(t))$, $t \in S_j$. Let $n_j(w) = n(g_j = w, S_j)$ be the number of roots in $S_j$ of the equation $g_j(t) = w$. Let

$$P_j(R) = \frac{1}{2\pi}\int_0^{2\pi} n_j(\mathrm{Re}^{i\theta})d\theta.$$

It is clear that $n_j(w) = n(f = w, D_j)$ and

$$\sum_{j=1}^{k} P_j(R) \le P(R) = P(R, D, f) = \frac{1}{2\pi}\int_0^{2\pi} n(f = \mathrm{Re}^{i\theta}, D)d\theta.$$

Set

$$\gamma_j(R) = \{t: |g_j(t)| = R, \quad t \in S_j\}, \quad \xi_1^{(j)}(R) = \inf\{\mathrm{Re}\, t: t \in \gamma_j(R)\},$$
$$\xi_2^{(j)}(R) = \sup\{\mathrm{Re}\, t: t \in \gamma_j(R)\}, \quad w_j(R) = \xi_2^{(j)}(R) - \xi_1^{(j)}(R).$$

Lemma 3 in [5, p. 153] then gives

$$\frac{1}{2}\int_{R_0}^{|f(re^{i\theta_j})|} \frac{dR}{RP_j(R)} = \log\frac{1}{1-r} + O(1), \quad r \to 1^-. \tag{2.1}$$

So it is easy to see that $g_j(t)$ satisfies the hypotheses of Lemma 1 in [4]. By (4.2) in [4, p. 105] we have

$$\xi_2^{(j)}(R_j) - \xi_1^{(j)}(R_0) \ge \frac{1}{2}\int_{R_0}^{R_j} \frac{1}{tP_j(t)}dt, \tag{2.2}$$

for some positive $R_0$ and all $R_j > R_0$. We put $R_j = |f(re^{i\theta_j})|$. Since $|g_j(t(re^{i\theta_j}))| = R_j$, $t(re^{i\theta_j}) \in \gamma_j(R_j)$, thus

$$\xi_1^{(j)}(R_j) \le \log\{r^{k/2}\prod_{j=1}^{q} |1 - re^{i(\theta_s - \theta_j)}|^{-1}\} \le \xi_2^{(j)}(R_j). \tag{2.3}$$

Now theorem 2 in [4, p. 108] shows that

$$\xi_s^{(j)}(R_j) - \frac{1}{2}\int_{R_0}^{R_j} \frac{1}{tP_j(t)}dt \to \beta, \quad r \to 1^-, \quad s = 1, 2, \tag{2.4}$$

where $-\infty < \beta \le +\infty$. From (2.1), (2.3) and (2.4) we obtain the following Lemma.

LEMMA 2.1: *If $f \in \mu(p, k)$, then $\lim_{R\to+\infty} w_j(R) = 0, j = 1, 2, \ldots, k$.*

THEOREM 2.1: *If $f \in \mu(p, k)$, then there exists a positive constant $A$ independent of $r$ such that*

$$(1-r)^{2pk}\prod_{j=1}^{k} |f(re^{i\theta_j})| \min_{1\le j\le k} |f(re^{i\theta_j})|^{k(k-1)} \le A < +\infty, \quad 0 < r < 1.$$

*Proof:* Set $R_j = |f(re^{i\theta_j})|$, $M^* = \min_{1\le j\le k} R_j$. Then (2.2) and (2.3) give

$$\begin{aligned}\log\{r^{\frac{k}{2}}\prod_{s=1}^{k}|e^{i\theta_s}-re^{i\theta_j}|^{-1}\} &\ge \xi_1^{(j)}(R_j)\\ &= \xi_2^{(j)}(R_j)-w_j(R_j)\\ &\ge \xi_1^{(j)}(R_0)+\frac{1}{2}\int_{R_0}^{R_j}\frac{dt}{tP_j(t)}-w_j(R_j).\end{aligned} \tag{2.5}$$

We sum (2.5) from $j=1$ to $k$ to find

$$\begin{aligned}&\log\{\prod_{1\le v<s\le k}|e^{i\theta_v}-re^{i\theta_s}|^{-2}\}+\log(1-r)^{-k}\\ &\ge \sum_{j=1}^{k}[\xi_1^{(j)}(R_0)-w_j(R_j)]+\frac{1}{2}\sum_{j=1}^{k}\int_{R_0}^{R_j}\frac{dt}{tP_j(t)}\\ &\ge \sum_{j=1}^{k}[\xi_1^{(j)}(R_0)-w_j(R_j)]+\frac{k^2}{2}\int_{R_0}^{M^*}\frac{dt}{t\sum_{j=1}^{k}P_j(t)}+\frac{1}{2}\sum_{j=1}^{k}\int_{M^*}^{R_j}\frac{dt}{tP_j(t)}\\ &\ge \sum_{j=1}^{k}[\xi_1^{(j)}(R_0)-w_j(R_j)]+\frac{k^2}{2}\int_{R_0}^{M^*}\frac{dt}{tP(t)}+\frac{1}{2}\sum_{j=1}^{k}\int_{M^*}^{R_j}\frac{dt}{tP(t)}\\ &= \sum_{j=1}^{k}[\xi_1^{(j)}(R_0)-w_j(R_j)]+\frac{k(k-1)}{2}\int_{R_0}^{M^*}\frac{dt}{tP(t)}+\frac{1}{2}\sum_{j=1}^{k}\int_{R_0}^{R_j}\frac{dt}{tP(t)},\end{aligned} \tag{2.6}$$

where we have used the inequality between arithmetic and harmonic means. From Lemma 2.1 in [6, p. 23], we have

$$\int_{R_0}^{R}\frac{dt}{tP(t)}\ge\frac{1}{p}\log\frac{R}{R_0}-\frac{1}{2p}. \tag{2.7}$$

Combining this with (2.6), we obtain

$$\begin{aligned}&\log(1-r)^{-k}+\log\{\prod_{1\le v<s\le k}|e^{i\theta_v}-re^{i\theta_s}|^{-2}\}\\ &\ge C(R_0,r)+\frac{k(k-1)}{2p}\log M^*+\frac{1}{2p}\sum_{j=1}^{k}\log R_j,\end{aligned} \tag{2.8}$$

where

$$C(R_0,r)=\sum_{j=1}^{k}[\xi_1^{(j)}(R_0)-w_j(R_j)]+\frac{k^2}{2p}\Big(\log\frac{1}{R_0}-\frac{1}{2}\Big).$$

Since $R_0$ is fixed, we see from Lemma 2.1 that $C(R_0, r)$ is bounded as $r$ tends to 1. The conclusion of Theorem 2.1 now follows from (2.8). This proof is complete.

THEOREM 2.2: *Suppose that $f(z)$ is areally mean $p$-valent in $D$. Then $f \in \mu(p,k)$ if and only if there exist $k$ distinct points $e^{i\theta_1}, \ldots, e^{i\theta_k}$ on $|z| = 1$, and there exist a constant $\delta > 0$ and a sequence $\{r_n\}$ with $r_n \to 1^-$ as $n \to \infty$ such that*

$$|f(r_n e^{i\theta_j})| \geq \delta(1 - r_n)^{-2p/k}, \quad j = 1, 2, \ldots, k, \tag{2.9}$$

*and*

$$M(r_n) \leq \frac{1}{\delta}(1 - r_n)^{-2p/k}, \tag{2.10}$$

*for all $n$.*

*Proof:* We only need to prove the necessity. From [5, p. 153], $f \in \mu(p,k)$ implies, in the notation of [4, p. 119] , that $f \in \mathcal{F}(k)$. Thus we have [4, p. 128]

$$\sup\{|f(z)|: z \in D_j, |z| = r\} \leq 2|f(re^{i\theta_j})| \leq 2M(r) \leq 4 \max_{1\leq j \leq k} |f(re^{i\theta_j})|, \tag{2.11}$$

for all $r$ sufficiently near 1. Taking $\{r_n\}$ and $\{\theta_n^{(j)}\}$ satisfying (1.1) and (1.2), we have $r_n e^{i\theta_n^{(j)}} \in D_j$, for all large $n$. Thus

$$\begin{aligned} 2|f(r_n e^{i\theta_j})| &> \sup\{|f(z)|: z \in D_j, |z| = r_n\} \\ &\geq |f(r_n e^{i\theta_n^{(j)}})| > c(1 - r_n)^{-2p/k}. \end{aligned} \tag{2.12}$$

We deduce from (2.11) and (2.12) that

$$(1 - r_n)^{2pk} \min_{1\leq j\leq k} |f(r_n e^{i\theta_j})|^{k(k-1)} \prod_{j=1}^{k} |f(r_n e^{i\theta_j})| \geq c_1 M(r_n)(1 - r_n)^{2p/k},$$

for all large $n$, where $c_1$ is a positive constant. Theorem 2.1 shows that the left-hand side is bounded by a constant independent of $n$, hence (2.10) holds. This completes the proof of Theorem 2.2.

## 3. On Bazilevic's Theorem

In this section, we shall denote by $c_1, c_2, \ldots$ any constants independent of $r$.

LEMMA 3.1: *Suppose that $f \in \mu(p,k)$ and that $z_1, \ldots, z_q$ are the zeros of $f$ in $D$. Then we have for $\theta \in [0, 2\pi]$,*

$$\begin{aligned} &|f(r_2 e^{i\theta})|(1-r_2)^{2p} \le e^{10p+\frac{1}{2}}|f(r_1 e^{i\theta})|(1-r_1)^{2p}, \\ &\tfrac{1}{2}(1 + \max_{1\le j\le q}|z_j|) < r_1 < r_2 < 1. \end{aligned} \tag{3.1}$$

*In particular,*

$$M(r) \le 4^p e^{10p+\frac{1}{2}} M(r^2), \quad \tfrac{1}{2}(1+\max_{1\le j\le q}|z_j|) < r^2 < 1. \tag{3.2}$$

*Proof:* From Lemma 2.4 in [6, p. 28], we have

$$\left|\int_{|f(r_1 e^{i\theta})|}^{|f(r_2 e^{i\theta})|} \frac{dR}{RP(R)}\right| \le 2\log\frac{1-r_1}{1-r_2} + 10, \quad \tfrac{1}{2}(1+\max_{1\le j\le q}|z_j|) < r_1 < r_2 < 1.$$

Combining this with (2.7), we obtain

$$\log\frac{|f(r_2 e^{i\theta})|}{|f(r_1 e^{i\theta})|} < 2p\log\frac{1-r_1}{1-r_2} + 10p + \tfrac{1}{2}.$$

This shows (3.1) is true and (3.2) follows easily by taking $r_1 = r_2^2 = r^2$ in (3.1).

LEMMA 3.2: *Suppose that $f \in \mu(p,k)$ and that $z_1, \ldots, z_q$ are the zeros of $f$ in $D$. Then, if $r_n$ is defined in Theorem 2.2, we have*

$$\int_{R_0}^{M(r_n)} \frac{H(R, r_0 \le |z| \le \sqrt{r_n})}{R^3} dR = O(1), \qquad \text{as } n \to \infty,$$

*where $r_0 = \frac{1}{4}(3 + \max_{1\le j\le q}|z_j|)$, $R_0$ is a fixed positive constant, and*

$$H(R, r_0 \le |z| < \sqrt{r_n}) = \frac{1}{\pi}\int\int_{|w|\le R} n(f = w, r_0 \le |z| < \sqrt{r_n}) du dv - pR^2,$$

$w = u + vi.$

*Proof:* Choose a fixed $t_0 \in (r_0, 1)$, let $r^2 \in (t_0, 1)$ and put $\xi_j = t_0 e^{i\theta_j}$, $f_j(\xi) = f(\xi_j + \delta\xi)$, $\xi \in D$, where $\delta = \frac{1}{2}(r + r^2) - t_0$. Then if $t_0$ is chosen near enough to 1, we see that

$$\sum_{j=1}^{k} P_j(R) \le P(R, r_0 \le |z| < r) = \frac{1}{2\pi}\int_0^{2\pi} n(f = \mathrm{Re}^{i\theta}, r_0 \le |z| < r) d\theta,$$

where

$$P_j(R) = \frac{1}{2\pi}\int_0^{2\pi} n(f_j = \mathrm{Re}^{i\theta}, D)d\theta.$$

Applying Theorem 2.2 in [6, p. 21] to $f_j(\xi)$, we obtain

$$\int_{|f(\xi_j)|}^{|f(r^2e^{i\theta_j})|} \frac{dR}{RP_j(R)} \leq 2\log\frac{1}{1-r^2} + C_2. \tag{3.3}$$

Set $r^2 = r_n, M_0 = \max_{1\leq j\leq k}|f(\xi_j)|$ and $M^* = \min_{1\leq j\leq k}|f(r^2e^{i\theta_j})|$; it follows from the inequality between arithmetic and harmonic means that

$$\begin{aligned}\int_{M_0}^{M^*} \frac{dR}{RP(R, r_0\leq|z|<r)} &\leq \int_{M_0}^{M^*}\frac{1}{k^2}\sum_{j=1}^k \frac{1}{RP_j(R)}dR\\ &\leq \frac{1}{k^2}\sum_{j=1}^k\int_{|f(\xi_j)|}^{|f(r^2e^{i\theta_j})|}\frac{dR}{RP_j(R)} \leq \frac{2}{k}\log\frac{1}{1-r^2} + C_3.\end{aligned} \tag{3.4}$$

From Lemma 2.1 in [6, p. 23] and Theorem 2.2, we have

$$\begin{aligned}&\int_{M_0}^{M^*}\frac{-H(R, r_0\leq|z|<r)}{p^2R^2}dR\\ &\leq \int_{M_0}^{M^*}\frac{dR}{RP(R, r_0\leq|z|<r)} - \frac{1}{p}\log M^* + C_4\\ &\leq \frac{2}{k}\log\frac{1}{1-r^2} - \frac{1}{p}\log M^* + C_5 \leq C_6.\end{aligned} \tag{3.5}$$

The fact that $n(f = w, r_0 \leq |z| < r)$ is non-negative gives $-H(R, r_0\leq|z|\leq r)\leq pR^2$. Hence, we get from Lemma 3.1 and Theorem 2.2 that

$$\begin{aligned}&\int_{R_0}^{M_0} + \int_{M^*}^{M(r_n)}\frac{-H(R, r_0\leq|z|<r_n)}{R^3}dR\\ &\leq p\log\frac{M_0}{R_0} + p\log\frac{M(r_n)}{M^*(r^2)} \leq C_7 + p\log\frac{C_8M(r_n)}{M^*(r_n)} \leq C_9.\end{aligned} \tag{3.6}$$

Lemma 3.2 follows directly from (3.5) and (3.6).

LEMMA 3.3: *Let the function $f$ be areally mean $p$-valent in $D$, and let $r_0$ be defined in Lemma 3.2. Set $m(r) = \min_{r_0\leq|z|\leq r}|f(z)|$. Then*

$$\begin{aligned}&\frac{1}{2\pi}\int\int_{|z|\leq r}\left|\left[\log\frac{f(z)}{h(z)}\right]'\right|^2 dxdy\\ &= p\log\frac{M(r)}{m(r)} + \int_{m(r)}^{M(r)}\frac{H(R, r_0\leq|z|\leq r)}{R^3}dR + B(r), \quad r_0\leq r<1,\end{aligned} \tag{3.7}$$

*where $B(r)$ is a bounded function in $[r_0, 1)$.*

*Proof:* By the residue theorem, we easily obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int\int_{r_0\le|z|\le r}\left|\left[\log\frac{f(z)}{h(z)}\right]'\right|^2 dxdy \\
&=\frac{1}{2\pi}\int\int_{r_0\le|z|\le r}\left|\frac{f'(z)}{f(z)}\right|^2 dxdy-\frac{1}{2}\sum_{j=1}^{q}\sum_{t=1}^{q}\log\frac{r^2-\bar{z}_j z_t}{r_0^2-\bar{z}_j z_t},
\end{aligned}\tag{3.8}$$

where $z_1, \ldots, z_q$ are zeros of $f(z)$ in $D$. From (6) and (7) in [3] (or (6) in [1]), we have

$$\begin{aligned}
&\frac{1}{2\pi}\int\int_{r_0\le|z|\le r}\left|\frac{f'(z)}{f(z)}\right|^2 dxdy \\
&=\int_{m(r)}^{M(r)}\frac{P(R, r_0\le|z|\le r)}{R}dR=\int_{m(r)}^{M(r)}\frac{p+h(R)}{R}dR \\
&=p\log\frac{M(r)}{m(r)}+\frac{1}{2}\int_{m(r)}^{M(r)}d\frac{H(R, r_0\le|z|\le r)}{R^2} \\
&\quad+\int_{m(r)}^{M(r)}\frac{H(R, r_0\le|z|\le r)}{R^3}dR.
\end{aligned}\tag{3.9}$$

Set

$$\begin{aligned}
B(r)=&\frac{1}{2\pi}\int\int_{|z|\le r_0}\left|\left[\log\frac{f(z)}{h(z)}\right]'\right|^2 dxdy \\
&+\frac{1}{2}\int_{m(r)}^{M(r)}d\frac{H(R, r_0\le|z|\le r)}{R^2}-\frac{1}{2}\sum_{j=1}^{q}\sum_{t=1}^{q}\log\frac{r^2-\bar{z}_j z_t}{r_0^2-\bar{z}_j z_t}.
\end{aligned}\tag{3.10}$$

Since $f$ is areally mean $p$-valent in $D$,

$$-p\le\frac{H(R, r_0\le|z|\le r)}{R^2}\le 0.$$

This shows that $B(r)$ is a bounded function in $[r_0, 1)$. From (3.8) to (3.10), we obtain easily (3.7) and the proof is complete.

THEOREM 3.1: *Suppose that $f\in\mu(p, k)$, and attains maximal growth on $t_j = e^{i\theta_j}$ $(j = 1, 2, \ldots, k)$. Then*

$$\sum_{m=1}^{\infty} m\left|c_m-\frac{1}{mk}\sum_{j=1}^{k}\bar{t}_j^m\right|^2<+\infty\tag{3.11}$$

*if and only if*

$$\int_0^{R_0}\left\{\frac{1}{R^3}\int\int_{|w|\le R} n(f=w, r_0\le |z|<1)dudv\right\}dR<+\infty, \tag{3.12}$$

*where $\{c_n\}$ are defined in (1.3), $r_0$ is defined in Lemma 3.2 and $R_0$ is a fixed positive number.*

It should be noted that under the hypotheses of Theorem A, we have $n(f=w, r_0\le|z|<1)=0$ for $|w|<A$. Thus, Theorem 3.1 certainly implies Theorem A.

*Proof:* From Lemma 3.3, we have

$$\begin{aligned}
I(r) &= \sum_{m=1}^{\infty} m|c_m - \frac{1}{mk}\sum_{j=1}^{k}\bar{t}_j^m|^2 r^{2m}\\
&= \sum_{m=1}^{\infty} m|c_m|^2 r^{2m} + \frac{1}{k^2}\sum_{m=1}^{\infty}\frac{1}{m}|\sum_{j=1}^{k}\bar{t}_j^m|^2 r^{2m} - \frac{2}{k}\sum_{j=1}^{k}\mathrm{Re}\sum_{m=1}^{\infty} c_m(r^2t_j)^m\\
&= \frac{1}{2p}\log\frac{M(r)}{m(r)} + \frac{1}{2p^2}\int_{m(r)}^{M(r)}\frac{H(R, r_0\le|z|<r)}{R^3}dR\\
&\quad + \frac{1}{k}\log\frac{1}{1-r^2} - \frac{1}{kp}\sum_{j=1}^{k}\log\left|\frac{f(r^2t_j)}{h(r^2t_j)}\right| + B_1(r)\\
&= \frac{1}{2p}\log\left\{(1-r^2)^{-2p/k}M(r)\prod_{j=1}^{k}|f(r^2t_j)|^{-2/k}\right\}\\
&\quad + \frac{1}{2p^2}\int_{m(r)}^{R_0}\frac{1}{R^3}\left\{\frac{1}{\pi}\int\int_{|w|\le R} n(f=w, r_0\le|z|<r)dudv\right\}dR\\
&\quad + \frac{1}{2p^2}\int_{R_0}^{M(r)}\frac{H(R, r_0\le|z|<r)}{R^3}dR + B_2(r),
\end{aligned}$$

where $B_1(r), B_2(r)$ are bounded functions in $[r_0, 1)$. Let $r_n$ be defined as in Theorem 2.2, and set

$$\begin{aligned}
S_n &= \frac{1}{2p}\log\{(1-r_n)^{-2p/k}M(\sqrt{r_n})\prod_{j=1}^{k}|f(r_nt_j)|^{-2/k}\}\\
&\quad + \frac{1}{2p^2}\int_{R_0}^{M(\sqrt{r_n})}\frac{H(R, r_0\le|z|<\sqrt{r_n})}{R^3}dR + B_2(r_n).
\end{aligned}$$

Then Lemmas 3.1, 3.2 and Theorem 2.2 show that $\{S_n\}$ is a bounded sequence. By Levi's Theorem,

$$\tag{3.13} \begin{aligned} &\lim_{r\to 1^-}\int_{m(r)}^{R_0}\left\{\frac{1}{R^3}\int\int_{|w|\le R} n(f=w, r_0\le |z|<r)dudv\right\}dR \\ &=\int_0^{R_0}\left\{\frac{1}{R^3}\int\int_{|w|\le R} n(f=w, r_0\le |z|<1)dudv\right\}dR. \end{aligned}$$

If we note that $I(r)$ is nondecreasing in $[r_0, 1)$, and that

$$I(\sqrt{r_n}) = S_n + \frac{1}{2\pi p^2}\int_{m(\sqrt{r_n})}^{R_0}\left\{\frac{1}{R^3}\int\int_{|w|\le R} n(f=w, r_0\le |z|\le \sqrt{r_n})dudv\right\}dR$$

for all large $n$, then from (3.13) we can complete the proof of Theorem 3.1.

## 4. An example

We now construct a function $g(z)$ in $\mu(p,k)$ that satisfies (3.12) but does not satisfy (1.4). Set

$$f_1(z) = \frac{1}{2}\frac{z}{(1-z)^2} + (\frac{1}{2}-\frac{i}{\pi})\frac{z}{1-z} + \frac{i}{2\pi}\frac{1+z}{1-z}\log\frac{1+z}{1-z}, \quad |z|<1.$$

Easy calculations show that $\mathrm{Re}\{(1-z)^2 f_1'(z)\} > 0$, $|z|<1$. Hence $f_1(z)$ is univalent in $D$. By considering $f_1(e^{i\theta})$, we see that $f_1(z)$ omits a disk $|w-w_0|<\epsilon$ for some $w_0\in\mathbb{C}$ and $\epsilon>0$. Put $f_2(z) = f_1(z)-w_0$, and let $G_\delta$ be a simply connected domain such that (i) $f_2(D)\subset G_\delta$, (ii) for all small $\rho>0$, $G_\delta\cap\{|w|<\rho\} = \{w=u+vi\colon u^2+v^2<\rho^2, 0<v<u^{1+\delta}, 0<u<\rho\}$, where $\delta>0$. By the Riemann mapping theorem, we see that there is a function of the form $f(z) = -w_0 + a_1z+\cdots$, $z\in D$, that maps $D$ univalently onto $G_\delta$. Since $f_2(z)$ is subordinate to $f(z)$, the Hayman index $\beta$ of $f$ cannot be smaller than that of $f_2$, so $\beta\ge\frac{1}{2}$. Thus $f\in\mu(1,1)$. Let $W(R)$ denote the area of the portion of $G_\delta$ lying in $|w|<R$. From (ii), we find $W(R)<R^{2+\delta}$ for all small $R$. We see that $f$ satisfies (3.12). Obviously, $f$ does not satisfy (1.4).

In general, for the function $g(z) = f(z^k)^{p/k}$, we get from Lemma 2 in [6, p. 95] that $g(z)$ is circumferentially mean $p$-valent in $D$. Since $f\in\mu(1,1)$, $g(z)\in\mu(p,k)$. Clearly, $g(z)$ satisfies (3.12) but does not satisfy (1.4).

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